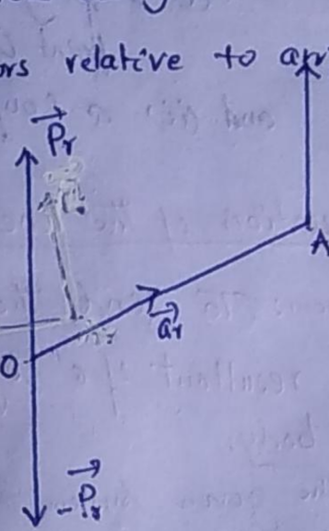
Study materials of Mathematics for Class D-III (H), Paper-VII on the topic "Coplanar forces acting on a rigid body" of Statics, Composed by Dr. S. Ahmed, Associate Professor

Reduction of a Coplanar System of forces

Theorem: Any System of Coplanar forces acting upon a rigid body can be reduced to a single force acting at an arbitrarily chosen point together with a couple.

Proof: Let $\vec{P_1}, \vec{P_2}, \dots \vec{P_n}$ be the Coplanar forces acting at the points $A_1, A_2, A_3, \dots, A_n$ whose position vectors relative to an origin O are $\vec{a_1}, \vec{a_2}, \dots \vec{a_n}$ respectively.



At O we introduce two forces $\vec{P_r}$ and $-\vec{P_r}$ in opposite directions parallel to $\vec{P_r}$ at A_r.

Since $\vec{P_r}$ and $-\vec{P_r}$ at O are equal in magnitude but opposite in directions. Hence they balance each other and will have no effect upon the given System.

Thus the forces $\vec{P_r}$ at A_r is equivalent to the following:

$\vec{P_r}$ at A_r, $\vec{P_r}$ at O & $-\vec{P_r}$ at O.

But the force $\vec{P_r}$ at A_r and the force $-\vec{P_r}$ at O form a Couple of moment $\vec{a_r} \times \vec{P_r}$

Hence the force $\vec{P_r}$ at A_r is equivalent to a parallel force $\vec{P_r}$ at O together with a couple of moment $\vec{a_r} \times \vec{P_r}$.

We now vary r, $(r = 1, 2, 3, \dots, n)$

Thus the given System of forces $\vec{P_1}, \vec{P_2}, \dots$ is equivalent to the parallel forces $\vec{P_1}, \vec{P_2}, \dots$ acting at O together with couples of moments $\vec{a_1} \times \vec{P_1}$, $\vec{a_2} \times \vec{P_2}$, ... Let the forces $\vec{P_1}, \vec{P_2}, \dots$

acting at O have resultant $\vec{R}$ so that

$$\vec{R} = \vec{P_1} + \vec{P_2} + \cdots = \sum_r \vec{P_r}$$

Also all the above couples can be compounded into a single couple of moment $\vec{G}$ such that

$$\vec{G} = \vec{a_1} \times \vec{P_1} + \vec{a_2} \times \vec{P_2} + \cdots = \sum_{r=1} \vec{a_r} \times \vec{P_r}.$$

Finally we see that the given system of forces $\vec{P_1}, \vec{P_2}, \ldots$ acting at points $A_1, A_2, \ldots$ respectively is reduced to

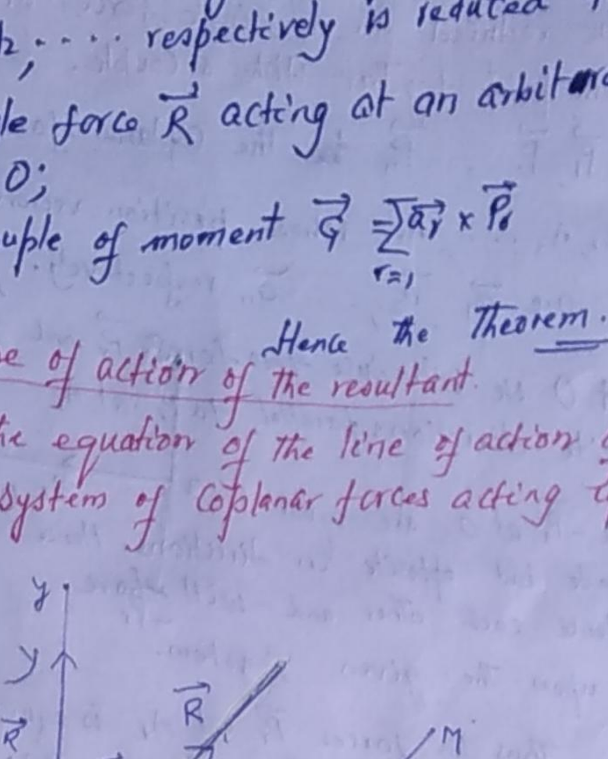
(i) a single force $\vec{R}$ acting at an arbitararily chosen point O;

and (ii) a couple of moment $\vec{G} = \sum_{r=1} \vec{a_r} \times \vec{P_r}$

Hence the Theorem.

Equation of the line of action of the resultant.

Theorem: To find the equation of the line of action of the resultant of a system of coplanar forces acting upon a rigid body.



Let the given system of coplanar forces be reduced to a single force $\vec{R}$ at the origin O together with a couple of moment $\vec{G}$.

The couple of moment $\vec{G}$ can be replaced by two equal and unlike parallel forces $-\vec{R}$ at O at a distance $\frac{G}{R}$ from O and parallel to the force $\vec{R}$ at O.

The forces $\vec{R}$ and $-\vec{R}$ at O balance each other and so the given system is reduced to a single resultant force $\vec{R}$ along a line LM at a distance $\frac{G}{R}$ from O and parallel to the original force at O

To find the ~~action~~ equation of the line of action LM of the resultant force.

Let LM cut the x-axis at the point A whose position vector is $\vec{a}$. Let P be any point on LM, whose position vector is $\vec{r}$.

Since $\vec{AP}$ is parallel to $\vec{R}$.

Hence $\vec{AP} = s\vec{R}$ where s is some scalar.

$$\Rightarrow \vec{OP} - \vec{PA} = s\vec{R} \Rightarrow \vec{r} - \vec{a} = s\vec{R}$$

$$\therefore \vec{r} = \vec{a} + s\vec{R} \quad \text{——— (1)}$$

Let $\hat{i}$ and $\hat{j}$ be unit vectors along the x-axis and the y-axis respectively. If X and Y are the components of $\vec{R}$ parallel to the x-axis and the y-axis respectively, then we can write

$$\vec{R} = X\hat{i} + Y\hat{j}$$

Also on taking moments about O

$$aY = G.$$

$$\therefore a = \frac{G}{Y}$$

Hence (1) becomes

$$\vec{r} = \left(\frac{G}{Y}\right)\hat{i} + s\vec{R} \quad [\vec{a} = a\hat{i}]$$

Thus the vector equation of the line of action LM of $\vec{R}$, the resultant, is

$$\boxed{\vec{r} = \left(\frac{G}{Y}\right)\hat{i} + s\vec{R}}$$

Cartesian Equivalent or Cartesian form

The above equation can be written as

$$x\hat{i} + y\hat{j} = \frac{G}{Y}\hat{i} + s(X\hat{i} + Y\hat{j})$$

Equating the coefficients of $\hat{i}, \hat{j}$, we have

$$x - \frac{G}{Y} = sX \quad \& \quad y = sY$$

(4)

Eleminating s, we have

$$x - \frac{G}{\gamma} = \frac{y}{\gamma} X$$

$$\boxed{x\gamma - yX = G}$$

It is the required equation of the resultant in the Cartesian form.